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Scientific Report No. 1
on
Contract NAS5-585

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National Aeronautics and Space Administration
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Report No. NAS 585-2
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Abstract

The input impedance of a cylindrical dipole in a homogeneous anisotropic ionosphere is determined for arbitrary values of the medium parameters and arbitrary orientation of the dipole with respect to the earth's magnetic field. A sinusoidal current distribution is assumed, as well as a low value of dipole excitation, so that the field equations may be assumed to be linear. The solution is obtained in the form of a rapidly converging integral which can be evaluated numerically without difficulty.

1. Introduction

For several years, ionosphere probing by rocket-borne probes has used the technique of the antenna impedance probe. This technique was first developed by NASA scientists [1]* and is now being pursued by other groups. It has been used principally in the lower regions of the ionosphere because of limitations imposed by the rocket. With the availability of more powerful rockets, it is now possible to carry probes of this type to altitudes above the maximum ionization levels of the ionosphere.

The impedance probe has the special merit that it is affected by the ionospheric parameters in the immediate vicinity of the probe — roughly in the order of a wavelength or less of the probe excitation. Consequently, this type of probe gives local values of the parameters, rather than integrated values as in the Seddon [2] type of experiment. It also can be operated at an excitation level low enough to avoid the creation of additional ionization in the medium by the r-f field.

The proper interpretation of the impedance measurement of an antenna probe requires a knowledge of the input impedance as a function of the medium properties. The type of antenna used in rocket probing can be rather closely approximated as a cylindrical dipole. Fortunately one is interested in the impedance change produced by the ionized medium, so that deviations from the ideal cylindrical form can be expected to play a secondary role. For this reason, the subject of this report is the calculation of the input impedance of a cylindrical dipole in a magneto-ionic medium.

Rocket experiments have shown that the antenna acquires a negative charge. This negative charge repels the electrons immediately around the antenna, with

*Numbers in brackets refer to the corresponding references in the Bibliography on p. 17.

the result that the antenna environment is rendered inhomogeneous. This effect is a very important one from the standpoint of the application of the impedance probe technique to the ionosphere. If this type of probe is to be useful, the nature of the inhomogeneity produced, and its effect on the antenna impedance must be capable of determination. The former of these two problems appears to be the more formidable one at present. In any event, a solution for the homogeneous case is a necessary first step, both to develop an insight to the nature and magnitude of the effects produced by various values of the medium parameters, as well as to serve as a basis for a possible perturbation technique for the inhomogeneous distribution. The case of a homogeneous medium will be assumed in this report.

The calculation of input impedance of a cylindrical dipole has been the subject of very extensive investigation for over 60 years. It has never been solved with complete rigor. But this is not meant to imply that the approximate results obtained are not useful. Discrepancies between calculated and measured values are large only when the dipole length approaches a wavelength or more. For most practical cases, and probably for all well-designed ionosphere probe experiments, the agreement with measurement is quite good, being better for the resistance component than the reactance component. The reactance component usually corresponds to that of a dipole whose length is slightly greater than the actual length. This may be viewed as an end effect equivalent to small capacitances at the ends of the dipole.

The calculation of input impedance of a dipole is a straightforward problem when the distribution of current over the dipole is known. The current distribution is not arbitrary, however, since, in principle, it can be derived from Maxwell's equations and the known boundary conditions at the dipole surface.

The key difficulty arises when one attempts to determine this current distribution from Maxwell's equations, since one is confronted with the problem of solving an integral equation. Various iterative methods have been employed, but the accuracy of the result, as judged by comparison with experiment, is sensitive to the technique used.

In first approximation the current distribution along the dipole is sinusoidal. This approximation is quite good for very thin, long dipoles. The sinusoidal distribution may be considered to be the result of guided waves propagating in the medium along the outside of the conductor and perfectly reflected at the open ends, the interference between the two oppositely directed wave trains resulting in a standing wave with zero current at the outer ends. Propagation along the wires takes place at the velocity appropriate to the external medium.

Fortunately, as already mentioned, in the case of a dipole used as an impedance probe we are not interested in the exact calculation of the dipole impedance, but rather in the impedance change upon entry into the ionosphere. This change is thus a difference quantity, so that small deviations in the nature of end effects which are occasioned by the use of only an approximate current distribution can be expected to largely cancel out when the difference, or change, from the free-space value is formed. Consequently one may simplify the problem immensely by assuming a sinusoidal current distribution. This will be done in the present treatment.

A further assumption is made that the amplitude of the motion of the free electrons in the medium in response to the electric field of the dipole is so small that the refractive index is given by the standard Appleton-Hartree formula. This assumption makes the field equations linear, so that Fourier resolutions are admissable.

Because of the complicated dependence of the refractive index of a magneto-ionic medium on the direction of propagation relative to the earth's magnetic field, it is customary in ionospheric propagation problems to make use of various approximations for the refractive index [3], depending on the directions of principal interest. In the antenna impedance problem, however, all directions of propagation are involved, so that the introduction of such simplifications is dangerous. Furthermore, in connection with ionosphere probing, virtually the whole gamut of normalized ionosphere parameters is of interest. For these reasons we have avoided the introduction of any such simplifications, in order to make the results obtained as widely applicable as possible.

In the treatment given in this report, therefore, we shall assume that the current is distributed sinusoidally along the dipole. The input impedance will be obtained by equating the complex power passing from the surface of the dipole into the medium to the complex power supplied to the dipole at its input terminals. The result will be reduced to an integral along a certain contour in the complex wave-direction plane. This integral can be evaluated numerically for the particular ionosphere parameters (plasma frequency, gyro frequency, collision frequency) of interest. The values of these parameters are not restricted in any way in the treatment. The orientation of the dipole with respect to the earth's magnetic field is arbitrary.

2. Formulation of the Problem

In view of the preceding discussion, we now undertake the calculation of the input impedance of a cylindrical dipole in an infinite homogeneous ionosphere having a constant superimposed magnetic field. The dipole, of radius r_0 and length 2ℓ, is considered to be fed at its center, and the current distribution will be taken to be symmetrical, and to be sinusoidal along each half of the dipole, being zero at the ends. This distribution is shown in Fig. 1.

We shall find it convenient to employ two coordinate systems, Σ and Σ'. Coordinate system Σ has its z-axis along the earth's magnetic field, $\underset{\sim}{H}_0$. For Σ', the z'-axis coincides with that of the dipole, and makes an angle $\theta \leq \pi/2$ with z. The relative orientations of Σ and Σ' is chosen so that the z'-axis lies in the yz-plane. Thus the x-axis of Σ and the x'-axis of Σ' coincide. The orientations of the two sets of axes are shown in Fig. 2.

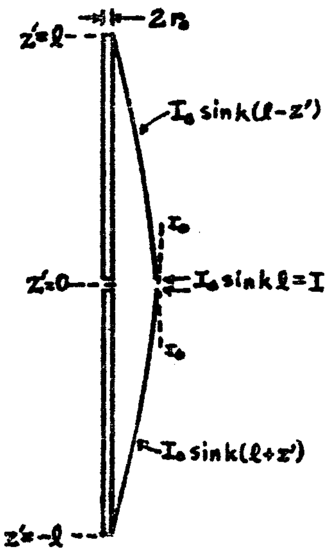

Fig. 1

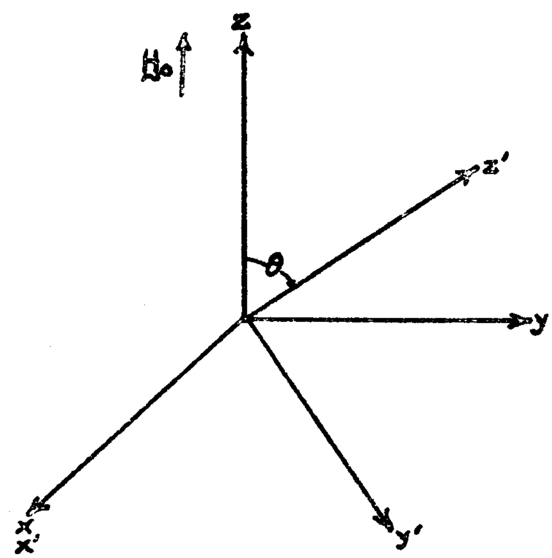

Fig. 2

We shall assume a time dependence of $e^{i\omega t}$ for all field quantities and sources. Rationalized m-k-s units will be used throughout.

The impedance will be determined by equating the complex power (i.e., volt-amperes) supplied to the antenna at its feed point to the complex power supplied by the dipole to the external medium. The latter is obtained by integrating the normal component of the complex Poynting vector over the surface of the dipole. For this we need to know the tangential electric and magnetic field strengths, $\underset{\sim}{E}$ and $\underset{\sim}{H}$, at the surface of the dipole due to the current in the dipole. Since the tangential component of $\underset{\sim}{H}$ can be related to the dipole current, the only problem is to find $\underset{\sim}{E}$.

The complex power supplied to the medium is

$$P = \frac{1}{2} \int_S (\underset{\sim}{E} \times \underset{\sim}{H}^*) \cdot d\underset{\sim}{S}, \tag{1a}$$

where S is the outer surface of the sipole and the direction of $d\underset{\sim}{S}$ is the outward normal. The complex power supplied to the dipole is

$$P = \frac{1}{2} V I^* = \frac{1}{2} |I^2| Z, \tag{1b}$$

where V and I are the input voltage and current, respectively, and Z is the input impedance. Consequently, on equating the two expressions for P we obtain

$$Z = \frac{1}{|I^2|} \int_S dP = \frac{1}{|I^2|} \int_S (\underset{\sim}{E} \times \underset{\sim}{H}^*) \cdot d\underset{\sim}{S}. \tag{2}$$

For the calculation of the integral in (2), we introduce cylindrical coordinates (r', φ', z') in Σ', where φ' is measured from the x'-axis. If we denote unit vectors by $\hat{e}$, then

$$d\underset{\sim}{S} = dA\,\hat{e}_{r'}.$$

The actual current flows on the surface $r' = r_0$. However, for the calculation of the fields we make the usual assumption that the current is a line source on the axis of the cylinder, $r' = 0$, and directed along the axis. Thus, if $j(z')$ represents this line current, then the actual surface density is

$$\underset{\sim}{K}(z') = \frac{1}{2\pi r_0} \underset{\sim}{j}(z') = \frac{j(z')}{2\pi r_0} \hat{e}_{z'}.$$

But the tangential component of $\underset{\sim}{H}$ at the surface is equal to the surface current density, so that

$$\begin{aligned} \underset{\sim}{H}_{tang}(r_0, \varphi', z') &= \frac{\underset{\sim}{j}(z') \times \hat{e}_{r'}}{2\pi r_0} = \frac{j(z')}{2\pi r_0} \hat{e}_{z'} \times \hat{e}_{r'} \\ &= \frac{j(z')}{2\pi r_0} \hat{e}_{\varphi'} . \end{aligned} \tag{3}$$

On inserting (3) into the integrand of (2), we obtain

$$(\underset{\sim}{E}\times\underset{\sim}{H}^*)\cdot\hat{e}_{r'} = \frac{1}{2\pi r_0} j^*(z')\,(\underset{\sim}{E}\times\hat{e}_{\varphi'})\cdot\hat{e}_{r'}$$

$$= \frac{1}{2\pi r_0} j^*(z')\ \underset{\sim}{E}\cdot(\hat{e}_{\varphi'}\times\hat{e}_{r'})$$

$$= -\frac{1}{2\pi r_0} j^*(z')\,\underset{\sim}{E}(r_0,\varphi',z')\cdot\hat{e}_{z'}\ . \tag{4}$$

But

$$\hat{e}_{z'} = \hat{e}_z\cos\theta + \hat{e}_y\sin\theta,$$

so that

$$\underset{\sim}{E}\cdot\hat{e}_{z'} = E_z\cos\theta + E_y\sin\theta.$$

Hence the outward component of the complex Poynting vector is

$$d\underset{\sim}{P} = \tfrac{1}{2}(\underset{\sim}{E}\times\underset{\sim}{H}^*)\cdot d\underset{\sim}{S} = -\frac{j^*(z')}{4\pi r_0}(E_z\cos\theta + E_y\sin\theta)r_0\,d\varphi' dz', \tag{5}$$

since $dA = r_0 d\varphi' dz'$. This has been expressed in the Σ frame, anticipating that the evaluation of the electric field will be simpler in this system than in Σ'. It should be noted that (5) does not involve E_x. This is due to the fact that the assumed axial current $j(z')$ has no component in the x-direction:

$$j(z') = j_z\cos\theta + j_y\sin\theta, \tag{6}$$

The problem thus reduces to a determination of the electric field $\underset{\sim}{E}$. Once this has been found, the values of E_y and E_z can be inserted in (5), and this then substituted in (2) to find the input impedance Z.

3. Calculation of E

We now turn to the problem of finding the electric field $\underset{\sim}{E}$. This obeys Maxwell's equations for the medium. In view of the free electrons and the superimposed static magnetic field of the earth, the medium is characterized by a dielectric tensor. This means that a given component of field is due to current components in all three coordinate directions. On the assumption that the field equations are linear (implying sufficiently small vibrations of the free charges), Maxwell's equations guarantee that a solution must exist for $\underset{\sim}{E}(\underset{\sim}{x})$ of the form

$$E_i(\underset{\sim}{x}) = \int G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)\, J_j(\underset{\sim}{x}_1)\, d^3x_1. \tag{7}$$

[At this point, a discussion of the notation to be employed is in order. $\underset{\sim}{x}$ denotes an arbitrary field point with coordinates x_x, x_y, x_z, $\underset{\sim}{x}_1$ a source point with coordinates x_{1x}, x_{1y}, x_{1z}, and $\ldots d^3x_1$ is a compact notation for $\ldots dx_{1x}dx_{1y}dx_{1z}$. The convention of summing over repeated indices will be used, so that, for example,

$$a_j b_j = \sum_{j=1}^{3} a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The repeated index thus is a dummy index, and free use will be made of changing the index letter where this is desirable.]

In (7), $G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ is the (i,j) component of the Green's tensor. Physically, it represents the electric field in the i^{th} direction at $\underset{\sim}{x}$ due to the j^{th} component of the source current density $\underset{\sim}{J}$ at $\underset{\sim}{x}_1$. Thus $G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ propagates the effect of $\underset{\sim}{J}$ at $\underset{\sim}{x}_1$ to the point $\underset{\sim}{x}$. Its existence is guaranteed by (actually, is a consequence of) the linearity of Maxwell's equations in $\underset{\sim}{E}$, $\underset{\sim}{H}$, $\underset{\sim}{J}$.

Since the derivation of the expression for $G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ is somewhat lengthy, it is given in Appendix A. The result is

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -\frac{i\omega\mu_0 k_a}{(2\pi)^3}\int d^3q\, \frac{\Delta_{ji}(q)}{\Delta(q)}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}, \tag{8}$$

from which (7) becomes

$$E_i(\underset{\sim}{x}) = -\frac{i\omega\mu_0 k_a}{(2\pi)^3}\int d^3x_1 \int d^3q\, \frac{\Delta_{ji}(q)}{\Delta(q)}\, J_j(\underset{\sim}{x}_1)\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}. \tag{9}$$

In (8) and (9), ρ is a numerical distance which is defined by

$$\underset{\sim}{\rho} = k_a(\underset{\sim}{x} - \underset{\sim}{x}_1), \tag{10}$$

k_a being the wave number which will appear in the current distribution.

Since (5) does not involve E_x, and (7) does not involve j_x, we need only four of the nine matrix elements of $G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ for the present problem. These are G_{22}, G_{23}, G_{32}, G_{33}. From (7) and $j_z = j_{z'}\cos\theta$, $j_y = j_{z'}\sin\theta$ we obtain

$$E_z = \int (G_{32} j_2 + G_{33} j_3)\, dx_1 = \int_{-\ell}^{\ell} dz_1''\, j(z_1'')\left[G_{32}(\underset{\sim}{x}'|\underset{\sim}{x}'')\sin\theta + G_{33}(\underset{\sim}{x}'|\underset{\sim}{x}'')\cos\theta\right]. \tag{11a}$$

$$E_y = \int (G_{22} j_2 + G_{23} j_3)\, dx_1 = \int_{-\ell}^{\ell} dz_1''\, j(z_1'') \left[G_{22}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') \sin\theta + G_{23}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') \cos\theta \right]. \qquad (11b)$$

4. Evaluation of Z

Having obtained the necessary expression for the electric field E, we can now proceed to the evaluation of the input impedance, Z. Inserting (11a) and (11b) into (5), we obtain

$$d\underset{\sim}{P} = -\frac{j^*(z')}{4\pi}\, dz'\, d\varphi' \int_{-\ell}^{\ell} dz_1''\, j(z_1'') \Big\{ \sin^2\theta\, G_{22}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + \cos^2\theta\, G_{33}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + \sin\theta\cos\theta \left[G_{23}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + G_{32}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') \right] \Big\}.$$

Hence, introducing this into (2), we obtain

$$Z = -\frac{1}{2\pi |I^2|} \int_{-\ell}^{\ell} dz' \int_0^{2\pi} d\varphi'\, j^*(z') \int_{-\ell}^{\ell} dz_1''\, j(z_1'') \Big\{ \sin^2\theta\, G_{22}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + \cos^2\theta\, G_{33}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + \sin\theta\cos\theta \left[G_{23}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') + G_{32}(\underset{\sim}{x}'|\underset{\sim}{x}_1'') \right] \Big\}. \qquad (12)$$

Again we introduce dimensionless variables by putting

$$\left.\begin{aligned} k_a z &= \zeta, \\ k_a \ell &= \Lambda. \end{aligned}\right\} \qquad (13)$$

Then (12) becomes

$$Z = -\frac{1}{2\pi k_a^2 |I^2|} \int_{-\Lambda}^{\Lambda} d\zeta' \int_0^{2\pi} d\varphi'\, j^*(\zeta') \int_{-\Lambda}^{\Lambda} d\zeta_1''\, j(\zeta_1'') \{ \sin^2\theta\, G_{22} + \cos^2\theta\, G_{33} + \sin\theta\cos\theta\, (G_{23} + G_{32}) \}. \qquad (14)$$

Formally, the final step is to substitute the required values of $G_{ij}(x'|x'')$ from (8) into (14) and perform the integrations to obtain the value for Z. The necessary combination of the G_{ij} involves the ratio $N(\underset{\sim}{q})/\Delta(\underset{\sim}{q})$, where

$$N(\underset{\sim}{q}) = \sin^2\theta\, \Delta_{22} + \cos^2\theta\, \Delta_{33} + \sin\theta\cos\theta\, (\Delta_{23} + \Delta_{32}),$$

or from M(q) as given in (A-21) of Appendix A,

$$N(\underset{\sim}{q}) = \sin^2\theta \left[(q^2 - \alpha_1)(q_2^2 - \alpha_3) + q_1^2(\alpha_3 - \alpha_1) \right] + \cos^2\theta \left[(q_3^2 - \alpha_1)(q^2 - \alpha_1) - \alpha_2^2 \right] + 2\sin\theta\cos\theta\, q_2 q_3 (q^2 - \alpha_1), \qquad (15)$$

where

$$q^2 = q_1^2 + q_2^2 + q_3^2,$$

and $\alpha_1, \alpha_2, \alpha_3$ are elements of the tensor κ which are expressed in terms of plasma frequency, gyro frequency, and collision frequency in (A-24a,b,c,) respectively.

Also, from (A-26) of Appendix A

$$\Delta(q) = \alpha_3 (q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2) \tag{16}$$

Consequently, using (8) and (16) in (14), we obtain

$$Z = \frac{i}{(2\pi)^2 |I|^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{k_0}{k_a} \int d^3q \frac{N(q)}{\alpha_3 (q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2)} \int_{-\Lambda}^{\Lambda} d\zeta' \int_{-\Lambda}^{\Lambda} d\zeta_1'' \int_0^{2\pi} d\varphi' j^*(\zeta') j(\zeta_1'') e^{-i q \cdot \ell} . \tag{17}$$

Note that (17) is of the proper dimensions, since $(j^* j)$ represents (current)^2 and $\sqrt{\frac{\mu_0}{\epsilon_0}}$ is the free-space impedance.

We now are ready to introduce the assumption of a sinusoidal current distribution. We express this distribution, which is illustrated in Fig. 1, in the form

$$\begin{aligned} j(\zeta) &= I_0 \sin(\Lambda - \zeta), \qquad & 0 \le \zeta \le \Lambda = k_a l \\ &= I_0 \sin(\Lambda + \zeta), \qquad & 0 \ge \zeta \ge -\Lambda \end{aligned} \tag{18}$$

and use this form for both $j^*(\zeta_1')$ and $j(\zeta_1'')$ in (17). It is then possible to carry out all the inner integrations in (17), which we denote by

$$\mathcal{L}_1(\underset{\sim}{q}) = \int_{-\Lambda}^{\Lambda} d\zeta' \int_{-\Lambda}^{\Lambda} d\zeta_1'' \int_0^{2\pi} d\varphi' j^*(\zeta') j(\zeta_1'') e^{-i q \cdot \ell} . \tag{19}$$

The quantity ρ is the numerical distance between the current element on the axis at ζ_1'' and the current element on the surface of the cylinder at ζ'. Hence

$$\underset{\sim}{q} \cdot \underset{\sim}{\rho} = q_{x'} (\zeta' - \zeta_1'') + q_{y'} R \sin\varphi' + q_{z'} R \cos\varphi'$$

where

$$R = k_a r_0 . \tag{20}$$

Then the exponent in (19) becomes

$$R(q_{x'} \cos\varphi' + q_{y'} \sin\varphi')$$

By putting

$$q_{x'} \cos\varphi' + q_{y'} \sin\varphi' = \sqrt{q_{x'}^2 + q_{y'}^2} \cos(\varphi' + \gamma)$$

$$\gamma = \tan^{-1} q_{y'}/q_{x'} ,$$

the innermost integral in (19) becomes, in view of the 2π-periodicity of the integrand

$$\int_0^{2\pi} e^{-iR(q_{x'}\cos\varphi' + q_{y'}\sin\varphi')}\, d\varphi' = 2\pi J_0\left(R\sqrt{q_{x'}^2 + q_{y'}^2}\right). \tag{21}$$

We may convert the q's from the Σ' to the Σ system by using

$$\hat{e}_{y'} = \hat{e}_y \cos\theta - \hat{e}_z \sin\theta.$$

The Bessel function in (21) then becomes

$$J_0\left(R\sqrt{q_1^2 + (q_2\cos\theta - q_3\sin\theta)^2}\right).$$

The evaluation of the remaining integrals in (19) is straightforward. The result for $\mathcal{S}_1(q)$ is

$$\mathcal{S}_1(q) = 8\pi |I_0^2| F(\underset{\sim}{q}),$$

where

$$F(\underset{\sim}{q}) = J_0\left(R\sqrt{q_1^2 + (q_2\cos\theta - q_3\sin\theta)^2}\right) \cdot \left\{\frac{\cos\Lambda - \cos[\Lambda(q_3\cos\theta + q_2\sin\theta)]}{(q_3\cos\theta + q_2\sin\theta)^2 - 1}\right\}^2. \tag{22}$$

Inserting this into (17) and noting from (18) that $I = j(0) = I_0 \sin\Lambda$ (see Fig. 1), we obtain

$$Z = \frac{i}{2\pi^3 \sin^2\Lambda} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{k_0}{k_a} \iiint_{-\infty}^{\infty} \frac{N(q)\, F(q)}{\alpha_3 (q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2)}\, dq_1 dq_2 dq_3 \tag{23}$$

The factor

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{k_0}{k_a} = \frac{\zeta_0}{n(\theta)},$$

where ζ_0 is the impedance of free space and $n(\theta)$ is the refractive index of the medium appropriate to the orientation of the cylinder axis (angle θ to the earth's magnetic field). Consequently the coefficient of the triple integral in (23) may be written as

$$C = \frac{i\zeta_0}{2\pi^3 n(\theta) \sin^2\Lambda}, \tag{24}$$

so that (23) becomes

$$Z = C \iiint_{-\infty}^{\infty} \frac{N(q)\, F(q)}{\alpha_3 (q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2)}\, dq_1 dq_2 dq_3. \tag{25}$$

The q-integrations will be handled by complex variable techniques. The q_3-integration is readily effected by residues. The q_1, q_2 integrations then will be transformed into cylindrical coordinates. These integrals, in general, will have to be evaluated numerically. This appears to be quite feasible on a

[illegible]

The [illegible] of [illegible] half-plane, while [illegible] lies are in the [illegible] plane. In order to perform the [illegible] properly, it is necessary to [illegible] the [illegible] of the integrand at infinity.

We first change the q_1, q_2 variables by the transformation

[illegible]

Then

[illegible]

(15) for $H(q)$ then becomes

[illegible] (20)

and (22) for $F(q)$ becomes

[illegible] (2[illegible])

where

[illegible] (2[illegible])

[illegible] (2[illegible])

[illegible]

[illegible]

[illegible]

Case [illegible]

[illegible]

Case [illegible]

[illegible]

dipoles, covers the entire range of dipole orientations except very close to $\theta = \pi/2$.

The J_0 and cosine functions converge at infinity only along the real axis. Consequently in Case (a) we split the $\cos\Lambda\xi$ in (27) into exponential terms, while J_0, when this controls the behavior at infinity, is split into the Hankel functions

$$J_0 = \tfrac{1}{2}(H_0^{(1)} + H_0^{(2)}).$$

This splits $F(\underset{\sim}{q})$ into a sum of terms, each of which has exponential behavior at infinity. Consequently the integration path for each term may be deformed into the appropriate half-plane and the result of the q_3-integration expressed as the sum of the residues at the poles (σ_1, σ_2 or $-\sigma_1, -\sigma_2$) in that half-plane.

If we denote the value of $N(\underset{\sim}{q})$ at the pole $q_3 = \pm\sigma_i$ (with $+\sigma_i$ in the lower half-plane) by $N_i^{\pm}$ and the corresponding values of ξ and η by $\xi_i^{\pm}$ and $\eta_i^{\pm}$, respectively, then (25) reduces to

$$Z = i\pi C\Big\{\left(1+\tfrac{1}{2}\cos 2\Lambda\right)\left(I_1^{+} + I_1^{-} - I_2^{+} - I_2^{-}\right) + \tfrac{1}{2}\left(I_3^{+} + I_3^{-} - I_4^{+} - I_4^{-}\right)$$

$$- \cos\Lambda\left(I_5^{+} + I_5^{-} - I_6^{+} - I_6^{-}\right)\Big\}, \qquad (30)$$

where

$$I_{\substack{1\\2}}^{\pm} = \int_0^{2\pi}\int_0^{\infty} \frac{N_i^{\pm}\, H_0^{(1)}(\eta_i^{\pm})}{D_i^{\pm}}\, \rho\, d\rho\, d\psi, \qquad (31a)$$

$$I_{\substack{3\\4}}^{\pm} = \int_0^{2\pi}\int_0^{\infty} \frac{N_i^{\pm}\, e^{\mp i 2\Lambda \xi_i^{\pm}} J_0(\eta_i^{\pm})}{D_i^{\pm}}\, \rho\, d\rho\, d\psi, \qquad (31b)$$

$$I_{\substack{5\\6}}^{\pm} = \int_0^{2\pi}\int_0^{\infty} \frac{N_i^{\pm}\, e^{\mp i \Lambda \xi_i^{\pm}} J_0(\eta_i^{\pm})}{D_i^{\pm}}\, \rho\, d\rho\, d\psi. \qquad (31c)$$

$$D_i^{\pm} = \alpha_i \sigma_i (\sigma_1^2 - \sigma_2^2)\left[(\xi_i^{\pm})^2 - 1\right]. \qquad (32)$$

In examining the behavior as $\rho \to \infty$, we find that

$$\xi_i^{\pm} \to \rho\left(\sin\psi \sin\theta \mp \sqrt{\tfrac{\ldots}{\ldots}}\cos\theta\right) \qquad (33a)$$

$$\xi_2^{\pm} \rightarrow p(\sin\psi\sin\theta \mp i\cos\theta), \qquad (33b)$$

$$\eta_1^{\pm} \rightarrow pR\left[1-\frac{\alpha_1}{\alpha_3}+\left(\sqrt{\frac{\alpha_1}{\alpha_3}}\cos\theta \pm i\sin\psi\sin\theta\right)^2\right]^{1/2} = R\left[1-\frac{\alpha_1}{\alpha_3}+\left(\pm i\xi_1^{\pm}\right)^2\right]^{1/2}, \qquad (33c)$$

$$\eta_2^{\pm} \rightarrow pR\left(\cos\theta \pm i\sin\theta\sin\psi\right) = \pm iR\xi_2^{\pm}, \qquad (33d)$$

$$N_1^{\pm} \rightarrow p^4\left(1-\frac{\alpha_1}{\alpha_3}\right)\left(\sin\psi\sin\theta \mp i\sqrt{\frac{\alpha_1}{\alpha_3}}\cos\theta\right)^2 = p^2\left(1-\frac{\alpha_1}{\alpha_3}\right)\left(\xi_1^{\pm}\right)^2, \qquad (33e)$$

$$N_2^{\pm} \rightarrow p^2\alpha_1\left[(\cos\theta \mp i\sin\psi\sin\theta)^2-\left(1-\frac{\alpha_3}{\alpha_1}\right)\sin^2\theta\cos^2\psi\right]. \qquad (33f)$$

Defining

$$f^{\pm} = \lim_{p=\infty}\left(\frac{1}{p^2}N_2^{\pm}\right) = \alpha_1\left[(\cos\theta \mp i\sin\psi\sin\theta)^2-\left(1-\frac{\alpha_3}{\alpha_1}\right)\sin^2\theta\cos^2\psi\right], \qquad (34a)$$

$$g_1^{\pm} = \lim_{p=\infty}\left(\frac{1}{p}\xi_1^{\pm}\right) = \sin\psi\sin\theta \mp i\sqrt{\frac{\alpha_1}{\alpha_3}}\cos\theta, \qquad (34b)$$

$$g_2^{\pm} = \lim_{p=\infty}\left(\frac{1}{p}\xi_2^{\pm}\right) = \sin\psi\sin\theta \mp i\cos\theta, \qquad (34c)$$

$$h_1^{\pm} = \lim_{p=\infty}\left(\frac{1}{p}\eta_1^{\pm}\right) = R\left[1-\frac{\alpha_1}{\alpha_3}+\left(\sqrt{\frac{\alpha_1}{\alpha_3}}\cos\theta \pm i\sin\psi\sin\theta\right)^2\right]^{1/2}, \qquad (34d)$$

$$h_2^{\pm} = \lim_{p=\infty}\left(\frac{1}{p}\eta_2^{\pm}\right) = R(\cos\theta \pm i\sin\psi\sin\theta) = \pm iRg_2^{\pm}, \qquad (34e)$$

the asymptotic forms of the integrals in (31a-c) then become

$$I_1^{\pm} \sim -i\left(\frac{\alpha_3}{\alpha_1}\right)^{1/2}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_1^{\pm}}\right)^{1/2}\frac{e^{\mp i p h_1^{\pm}}}{p^2(g_1^{\pm})^2}\,dp\,d\psi, \qquad (35a)$$

$$I_2^{\pm} \sim -\frac{i}{\alpha_3-\alpha_1}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_2^{\pm}}\right)^{1/2}\frac{f^{\pm}e^{\mp i p h_2^{\pm}}}{p^4(g_2^{\pm})^4}\,dp\,d\psi, \qquad (35b)$$

$$I_3^{\pm} \sim -i\left(\frac{\alpha_3}{\alpha_1}\right)^{1/2}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_1^{\pm}}\right)^{1/2}\frac{e^{\mp i2\Lambda p g_1^{\pm}}}{p^2(g_1^{\pm})^2}\cos(ph_1^{\pm}-\pi/4)\,dp\,d\psi, \qquad (35c)$$

$$I_4^{\pm} \sim -\frac{i}{\alpha_3-\alpha_1}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_2^{\pm}}\right)^{1/2}\frac{f^{\pm}e^{\mp i2\Lambda p g_2^{\pm}}}{p^4(g_2^{\pm})^4}\cos(ph_2^{\pm}-\pi/4)\,dp\,d\psi, \qquad (35d)$$

$$I_5^{\pm} \sim -i\left(\frac{\alpha_3}{\alpha_1}\right)^{1/2}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_1^{\pm}}\right)^{1/2}\frac{e^{\mp i\Lambda p g_1^{\pm}}}{p^2(g_1^{\pm})^2}\cos(ph_1^{\pm}-\pi/4)\,dp\,d\psi, \qquad (35e)$$

$$I_6^{\pm} \sim -\frac{i}{\alpha_3-\alpha_1}\int_0^{2\pi}\int_0^{\infty}\left(\frac{2}{\pi p h_2^{\pm}}\right)^{1/2}\frac{f^{\pm}e^{\mp i\Lambda p g_2^{\pm}}}{p^4(g_2^{\pm})^4}\cos(ph_2^{\pm}-\pi/4)\,dp\,d\psi. \qquad (35f)$$

In each of these integrals, the exponential has a negative real part. This, together with the factor $p^{-5/2}$, insures that the integrands decay rapidly, so that only a restricted part of the p-interval contributes to the value of the integral. Consequently (30) is in a satisfactory form for numerical evaluation.

Comparison with Free-Space Solution

A comparable form of solution may be obtained for the free-space case by starting with the free-space Green's tensor given in (A-33) of Appendix A:

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -\frac{i\omega\mu_0 k_0}{(2\pi)^3}\int d^3q \, \frac{q_j q_i - \delta_{ji}}{q^2-1} e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\ell}} .$$

From this, the expression in braces in (14) is found to be

$$\sin^2\theta \, G_{22} + \cos^2\theta \, G_{33} + \sin\theta\cos\theta \, (G_{23}+G_{32})$$
$$= -\frac{i\omega\mu_0 k_0}{(2\pi)^3}\int d^3q \, \frac{G(\underset{\sim}{q})}{q^2-1} e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\ell}},$$

where

$$G(\underset{\sim}{q}) = (q_2^2-1)\sin^2\theta + (q_3^2-1)\cos^2\theta + 2q_2q_3\sin\theta\cos\theta. \tag{36}$$

From this, since the ζ_1'', φ', ζ' integrals are exactly the same as before, we obtain instead of (25)

$$Z_0 = C_0 \iiint_{-\infty}^{+\infty} \frac{G(\underset{\sim}{q})F(\underset{\sim}{q})}{q^2-1} dq_1 dq_2 dq_3 \tag{37}$$

where

$$C_0 = \frac{k_a}{k_0} C = \frac{i\zeta_0}{2\pi^3\sin^2\Lambda} . \tag{38}$$

Consequently, in order that (25) reduce to (37) in the free-space limit (x = 0) we must show that

$$\lim_{x=0} \frac{N(\underset{\sim}{q})}{\alpha_3(q_3^2-\sigma_1^2)(q_3^2-\sigma_2^2)} = \frac{G(\underset{\sim}{q})}{q^2-1} \tag{39}$$

As $x \to 0$, we see from (A-24a-c), (A-25) and (A-29) that

$$\alpha_1 \to 1, \; \alpha_2 \to 0, \; \alpha_3 \to 1, \quad \sigma_{\frac{1}{2}}^2 \to 1-p^2,$$

so that

$$\lim_{x=0}(q_3^2-\sigma_1^2) = \lim_{x=0}(q_3^2-\sigma_2^2) = q_3^2 - 1 + p^2 = q^2-1.$$

Also from (15) we find

$$\lim_{x=0} N(\underset{\sim}{q}) \equiv {}_0N(\underset{\sim}{q}) = (q^2-1)\left[(q_3\cos\theta + q_2\sin\theta)^2 - 1\right].$$

Hence the left-hand side of (39) reduces to

$$\frac{{}_0N(\underset{\sim}{q})}{(q^2-1)^2} = \frac{(q_3\cos\theta + q_2\sin\theta)^2 - 1}{q^2-1} \tag{40}$$

A rearrangement of (36) gives

$$\begin{aligned} G(\underset{\sim}{q}) &= q_2^2\sin^2\theta + q_3^2\cos^2\theta + 2q_2q_3\sin\theta\cos\theta - 1 \\ &= (q_2\sin\theta + q_3\cos\theta)^2 - 1, \end{aligned} \tag{41}$$

so that the right-hand side of (39) gives the same result as (40). This demonstrates that the expression obtained for Z reduces properly in the free-space case.

The expression (37) for the free-space impedance can be integrated exactly in terms of exponential integrals, and then separated into its real and imaginary parts to give the familiar expressions for input resistance and reactance in terms of sine and cosine integrals. The usual procedure in obtaining these relations is not from (37), but to start from the integrated form (A-33) of Appendix A and integrate this over the current distribution.

It was mentioned in the Introduction that in ionosphere probing we are interested in the change in impedance of the dipole on entry into the ionosphere. Consequently we may form the difference between the expressions given in (30) and (37), and calculate the differential impedance itself numerically. For this purpose (37) is first integrated with respect to q_3 in the same manner as (25), after first transforming q_1, q_2 to the p, ψ variables. This gives for Z_0 the same form as given in (30), but with the quantities N and D replaced by $G(\underset{\sim}{q})$ as given in (41) and

$${}_0D = (1-p^2)^{1/2}(\xi^2-1), \tag{42}$$

respectively.

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Appendix A

Calculation of the Green's Tensor

We calculate the $G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ of (7) from Maxwell's equations. To do this, we eliminate $\underset{\sim}{H}$ from the two coupled first-order differential equations to obtain a second-order differential equation for $\underset{\sim}{E}$. The inverse of this second-order differential equation then leads to the Green's tensor.

The first-order equations are

$$\text{curl } \underset{\sim}{E} = -i\omega\mu_0 \underset{\sim}{H}, \tag{A-1}$$

$$\text{curl } \underset{\sim}{H} = \underset{\sim}{J} + i\omega\epsilon_0 \bar{\kappa}\cdot\underset{\sim}{E}, \tag{A-2}$$

where $\bar{\kappa}$ is the dielectric tensor $\{\kappa_{ij}\}$ and J is the source current density. Eliminating H, we obtain

$$\text{curl curl } \underset{\sim}{E} = -i\omega\mu_0 \,\text{curl}\, \underset{\sim}{H} = -i\omega\mu_0 \underset{\sim}{J} + \frac{\omega^2}{c^2}\bar{\kappa}\cdot\underset{\sim}{E},$$

or, using curl curl = grad div $-\nabla^2$,

$$\text{grad div } \underset{\sim}{E} - \nabla^2 \underset{\sim}{E} - k_0^2 \bar{\kappa}\cdot\underset{\sim}{E} = -i\omega\mu_0 \underset{\sim}{J}, \tag{A-3}$$

where $k_0 = \omega/c$ = free-space wave number. $\nabla^2 E$ is understood to represent the Laplacian operating on $\underset{\sim}{E}$ only in rectangular coordinates. In these coordinates, (A-3), written in component form, is

$$\left(\partial_j\partial_i - \nabla^2\delta_{ji} - k_0^2\kappa_{ji}\right)E_i(\underset{\sim}{x}) = -i\omega\mu_0 J_j(\underset{\sim}{x}) \tag{A-4}$$

where

$$\partial_i = \frac{\partial}{\partial x_i}, \qquad \delta_{ji} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Since (A-3) is linear in $\underset{\sim}{E}$ and $\underset{\sim}{J}$, we must be able to write $\underset{\sim}{E}$ and $\underset{\sim}{J}$ as linear functionals of each other. (A-4) gives $\underset{\sim}{J}$ as a linear functional of $\underset{\sim}{E}$, but what we need is $\underset{\sim}{E}$ as a linear functional of $\underset{\sim}{J}$. Since (A-4) involves differential operations on $\underset{\sim}{E}$, it follows that $\underset{\sim}{E}$ must be an integral operator on $\underset{\sim}{J}$. Thus we must be able to write

$$E_i(\underset{\sim}{x}) = \int G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) J_j(\underset{\sim}{x}_1)\, d^3x_1 \,. \tag{A-5}$$

Inserting (A-5) into (A-4), we obtain

$$\int (\partial_i \partial_j - \nabla^2 \delta_{ij} - k_o^2 \kappa_{ij}) G_{jk}(\underset{\sim}{x}|\underset{\sim}{x}_1) J_k(\underset{\sim}{x}_1)\, d^3x_1 = -i\omega\mu_o J_i(\underset{\sim}{x}). \tag{A-6}$$

We can write (A-6) in a more transparent manner by using the properties of the Dirac delta function $\delta(\underset{\sim}{x}-\underset{\sim}{x}_1)$. These properties are summarized by

$$\int \delta(\underset{\sim}{x}-\underset{\sim}{x}_1)\, f(\underset{\sim}{x}_1)\, d^3x_1 = f(\underset{\sim}{x}).$$

Then

$$J_i(\underset{\sim}{x}) = \int \delta(\underset{\sim}{x}-\underset{\sim}{x}_1) J_i(\underset{\sim}{x}_1)\, d^3x_1 = \int \delta_{ik}\, \delta(\underset{\sim}{x}-\underset{\sim}{x}_1) J_k(\underset{\sim}{x}_1)\, d^3x_1 .$$

With this, (A-6) becomes

$$\int (\partial_i \partial_j - \nabla^2 \delta_{ij} - k_o^2 \kappa_{ij}) G_{jk}(\underset{\sim}{x}|\underset{\sim}{x}_1) J_k(\underset{\sim}{x}_1)\, d^3x_1 = -i\omega\mu_o \int \delta_{ik}\, \delta(\underset{\sim}{x}-\underset{\sim}{x}_1) J_k(\underset{\sim}{x}_1)\, d^3x_1 . \tag{A-7}$$

From this it is apparent that $G_{jk}(\underset{\sim}{x}|\underset{\sim}{x}_1)$ is given by

$$(\partial_i \partial_j - \nabla^2 \delta_{ij} - k_o^2 \kappa_{ij}) G_{jk}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -i\omega\mu_o \delta_{ik}\, \delta(\underset{\sim}{x}-\underset{\sim}{x}_1). \tag{A-8}$$

(A-8) still requires the inversion of a differential operator in order to obtain G. A common technique to use in cases of this kind is to Fourier analyze G. This accomplishes the same end as the use of operational calculus, since differentiation leads to multiplication, whose inverse, division, is known. To this end we set

$$G_{jk}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -i\omega\mu_o \int d^3k \int d^3k_1\, g_{jk}(\underset{\sim}{k},\underset{\sim}{k}_1)\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, e^{-i\underset{\sim}{k}_1\cdot\underset{\sim}{x}_1}, \tag{A-9}$$

and

$$\delta(\underset{\sim}{x}-\underset{\sim}{x}_1) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_1)} . \tag{A-10}$$

(A-10) follows from the completeness and orthonormality of $(2\pi)^{-3/2} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}$. On insertion of (A-9) and (A-10) into (A-8) we obtain

$$\begin{aligned} -i\omega\mu_o \int d^3k(-k_i k_j + k^2\delta_{ij} - k_o^2\kappa_{ij})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} \int d^3k_1\, g_{jk}(\underset{\sim}{k},\underset{\sim}{k}_1) e^{-i\underset{\sim}{k}_1\cdot\underset{\sim}{x}_1} \\ = -\frac{i\omega\mu_o}{(2\pi)^3}\, \delta_{ik} \int d^3k\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}_1}. \end{aligned} \tag{A-11}$$

Since the functional dependence on x_1 must be the same on both sides of this equation, it follows that we can write

$$\int d^3k_1\, g_{jk}(\underset{\sim}{k},\underset{\sim}{k}_1)\, e^{-i\underset{\sim}{k}_1\cdot\underset{\sim}{x}_1} = \frac{1}{(2\pi)^3}\, e^{i\underset{\sim}{k}\cdot x_1}\, f_{jk}(k), \tag{A-12}$$

where $f_{jk}(k)$ is some as yet unspecified function of k. Then it is evident from

$$\overline{K} = \begin{pmatrix} \alpha_1 & -i\alpha_2 & 0 \\ i\alpha_2 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{A-23}$$

where, with $u = 1 - iz$,

$$\alpha_1 = 1 - \frac{xy}{u^2 - y^2} \tag{A-24a}$$

$$\alpha_2 = \frac{xy}{u^2 - y^2} \tag{A-24b}$$

$$\alpha_3 = 1 - \frac{x}{u} \tag{A-24c}$$

x,y,z are the usual normalized (plasma frequency)^2, gyro frequency, and collision frequency, respectively:

$$\left.\begin{aligned} x &= (\omega_N/\omega)^2, \\ y &= \omega_H/\omega \ , \\ z &= \nu/\omega \ . \end{aligned}\right\} \tag{A-25}$$

(A-25)

Then

$$\Delta(\underset{\sim}{q}) = \det(M(\underset{\sim}{q}) = \alpha_3(q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2), \tag{A-26}$$

where σ_1^2 and σ_2^2 are the roots of the bi-quadratic

$$\alpha_3\sigma^4 - [2\alpha_1\alpha_3 - (\alpha_1 + \alpha_3)p^2]\sigma^2 + (p^2 - \alpha_3)(\alpha_1 p^2 - \alpha_1^2 - \alpha_2^2) = 0, \tag{A-27}$$

with

$$p^2 = q_1^2 + q_2^2. \tag{A-28}$$

These roots thus are given by

$$\sigma_{\frac{1}{2}}^2 = \frac{1}{2\alpha_3}\left\{2\alpha_1\alpha_3 - (\alpha_1 + \alpha_3)p^2 \pm \sqrt{(\alpha_3 - \alpha_1)^2 p^4 + 4\alpha_3\alpha_2^2 p^2 - 4\alpha_2^2\alpha_3^2}\right\} \tag{A-29}$$

Inserting (A-26) into (A-19), we obtain finally

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_0) = -\frac{i\omega\mu_0 k_a}{(2\pi)^3}\int d^3q \, \frac{\Delta_{ji}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\ell}}}{\alpha_3(q_3^2 - \sigma_1^2)(q_3^2 - \sigma_2^2)}. \tag{A-30}$$

Reduction to Free-Space Green's Tensor

In the case of free space, $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 0$, so that (A-29) reduces to

$$\sigma_{\frac{1}{2}}^2 = 1 - p^2.$$

Hence

comparison of the two sides of (A-11) that

$$(k^2\delta_{ij} - k_0^2\kappa_{ij} - k_i k_j) f_{jk}(\underset{\sim}{k}) = \delta_{ik}. \tag{A-13}$$

(A-8) is a system of algebraic equations in $f_{jk}(k)$. These can be inverted by Kramer's rule, or by ordinary matrix inversion. Thus if we define

$$M_{ij} \equiv k^2\delta_{ij} - k_i k_j - k_0^2\kappa_{ij}, \tag{A-14}$$

then

$$M_{ij}^{-1} = \frac{\Delta_{ji}}{\Delta}, \tag{A-15}$$

where Δ_{ji} is the $(j,i)^{th}$ cofactor of M and Δ = det.M. Hence from (A-13)

$$f_{jk}(\underset{\sim}{k}) = \frac{\Delta_{kj}}{\Delta}. \tag{A-16}$$

Then from (A-9), (A-12), (A-16), we obtain

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -\frac{i\omega\mu_0}{(2\pi)^3}\int d^3k\,\frac{\Delta_{ji}(\underset{\sim}{k})}{\Delta(\underset{\sim}{k})}e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_1)}. \tag{A-17}$$

We now introduce the dimensionless variables

$$\left.\begin{aligned} \underset{\sim}{k}/k_a &= \underset{\sim}{q}, \\ k_a(\underset{\sim}{x}-\underset{\sim}{x}_1) &= \underset{\sim}{\rho}. \end{aligned}\right\} \tag{A-18}$$

Then (A-17) becomes

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -\frac{i\omega\mu_0 k_a}{(2\pi)^3}\int d^3q\,\frac{\Delta_{ji}(\underset{\sim}{q})}{\Delta(\underset{\sim}{q})}e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}, \tag{A-19}$$

Substitution of (A-19) into (A-5) gives, finally

$$E_i(\underset{\sim}{x}) = -\frac{i\omega\mu_0 k_a}{(2\pi)^3}\int d^3x_1\int d^3q\,\frac{\Delta_{ji}(\underset{\sim}{q})}{\Delta(\underset{\sim}{q})}J_j(\underset{\sim}{x}_1)\,e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}. \tag{A-20}$$

From (A-14), the matrix M in the dimensionless coordinates q is

$$M(\underset{\sim}{q}) = \begin{pmatrix} q^2 - q_1^2 - \kappa_{11} & -q_1q_2 - \kappa_{12} & -q_1q_3 - \kappa_{13} \\ -q_2q_1 - \kappa_{21} & q^2 - q_2^2 - \kappa_{22} & -q_2q_3 - \kappa_{23} \\ -q_3q_1 - \kappa_{31} & -q_3q_2 - \kappa_{32} & q^2 - q_3^2 - \kappa_{33} \end{pmatrix} \tag{A-21}$$

where

$$q^2 = q_1^2 + q_2^2 + q_3^2. \tag{A-22}$$

In the coordinate system Σ, κ takes on the particularly simple form

Since for free space $\kappa_{ii}=1$, $\kappa_{ij}=0$, we obtain from (A-21)

$$\Delta_{ji} = q_j q_i (q^2-1), \qquad i \neq j \tag{A-31}$$

$$\Delta_{ii} = (q_i^2 - 1)(q^2-1), \tag{A-32}$$

while from (A-26)

$$\Delta = (q_3^2 - 1 + p^2)^2 = (q^2-1)^2.$$

By writing

$$q_i q_j = q_i q_j - \delta_{ij}$$

(A-31) and (A-32) may be written in the single form

$$\Delta_{ji} = (q_j q_i - \delta_{ji})(q^2-1).$$

Hence (A-30) becomes

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = -\frac{i\omega\mu_0 k_0}{(2\pi)^3}\int d^3q \, \frac{q_j q_i - \delta_{ji}}{q^2-1}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}.$$

Noting that $q_k = \dfrac{\partial}{\partial(-i\rho_k)} = i\dfrac{\partial}{\partial\rho_k}$, we may replace $q_j q_i$ by $-\dfrac{\partial}{\partial\rho_j}\dfrac{\partial}{\partial\rho_i}$, and obtain

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = \frac{i\omega\mu_0 k_0}{(2\pi)^3}\left(\frac{\partial}{\partial\rho_i}\frac{\partial}{\partial\rho_j} + \delta_{ij}\right)\int d^3q \frac{e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}}{q^2-1}.$$

The permutation of the indices is allowable because of the isotropy of $\underset{\sim}{q}\cdot\underset{\sim}{\rho}$.

We evaluate the q-integral by transforming to polar coordinates:

$$d^3q = q^2\,dq\,d\varphi\,\sin\theta\,d\theta$$

$$e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}} = e^{-iq\rho\cos\theta}$$

so that

$$\int d^3q\frac{e^{-i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}}{q^2-1} = \int_0^\infty \frac{q^2dq}{q^2-1}\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, e^{-iq\rho\cos\theta}\,d\theta$$

$$= \frac{2\pi i}{\rho}\int_0^\infty \frac{q\,dq}{q^2-1}\left(e^{-iq\rho} - e^{iq\rho}\right) = \frac{2\pi i}{\rho}\int_{-\infty}^{\infty}\frac{q\,e^{-iq\rho}}{q^2-1}dq.$$

The last integral has poles at q = ± 1. We detour around these in the usual manner as shown in Fig. 3, which is equivalent to assuming that q has a small negative imaginary part. Then, by deforming the path of integration into an infinite semicircle in the lower half-plane, we obtain $-2\pi i\cdot$(residue at q = 1), which is

$$-\frac{2\pi i}{2}e^{-i\rho}.$$

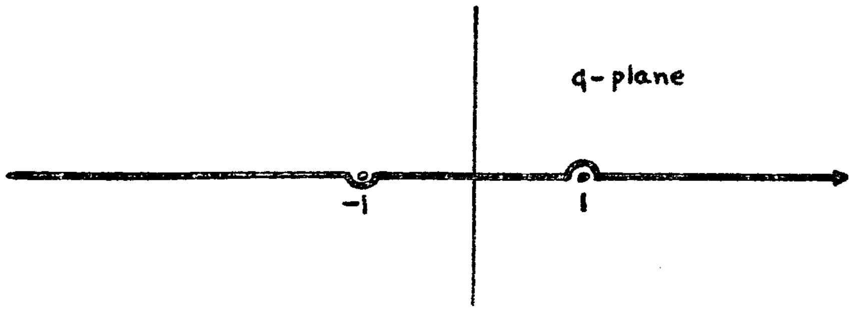

Fig. 3

Thus we obtain for G_{ij}

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = \frac{i\omega\mu_0 k_0}{(2\pi)^3}\left(\frac{\partial^2}{\partial\rho_i\partial\rho_j} + \delta_{ij}\right)\frac{e^{-i\rho}}{\rho}$$

which, on putting $\rho = k_0 r$, becomes

$$G_{ij}(\underset{\sim}{x}|\underset{\sim}{x}_1) = \frac{i\omega\mu_0}{4\pi}\left(\frac{1}{k_0^2}\frac{\partial^2}{\partial x_i\partial x_j} + \delta_{ij}\right)\frac{e^{-ik_0 r}}{r} \qquad \text{(A-33)}$$

which is the form of the Green's tensor for free-space.